

Propagation of gravitational waves from slow motion sources in a Coulomb type potential

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We consider the propagation of gravitational waves generated by slow motion sources in Coulomb type potential due to the mass of the source. Then, the formula for gravitational waveform including tail is obtained in a straightforward manner by using the spherical Coulomb function. We discuss its relation with the formula in the previous work.

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I. INTRODUCTION

It becomes urgent to study the generation and the propagation of gravitational waves in detail, because of the increasing expectation of direct detection of gravitational waves by Kilometer-size interferometric gravitational wave detectors, such as LIGO [1] and VIRGO [2] now under construction. Coalescing binary neutron stars are the most promising candidate of sources of gravitational waves for such detectors. Gravitational waves generated by them bring us informations not only on various physical parameters of neutron star [3,4], but also on the cosmological parameters [5–8] if and only if we can make a detailed comparison between the observed signal with theoretical prediction during the epoch of the so-called inspiraling phase where the orbital separation is much larger than the radius of component stars [9]. The problem is that, in order to make any successful comparison between theory and observation, we need to know the detailed waveform generated by the motion up to 4PN order [10] which is of order ϵ^8 higher than the Newtonian order, where $\epsilon = \text{orbital velocity}/\text{light speed}$.

Blanchet and Damour have developed a systematic scheme to calculate the waveform at higher orders, where the post-Minkowskian approximation is used to construct the external field and the post-Newtonian approximation is used to construct the field near the material source [11–13]. In the post-Minkowskian approximation, the background geometry is the Minkowski spacetime where linearized gravitational waves propagate [14,11,12]. The corrections to propagation of gravitational waves can be taken into account by performing the post-Minkowskian approximation up to higher orders. In fact, Blanchet and Damour obtained the tail term of gravitational waves as the integral over the past history of the source [12,13]. They introduced a complex parameter B [11], and used the analytic continuation as a useful mathematical device to evaluate the so-called log term in the tail contribution [12,13]. The method is powerful, but it is not easy to see the origin of the tail term. Will and Wiseman [15] have also obtained the tail term associated with the mass quadrupole moment, by generalizing the Epstein-Wagoner formalism [16]. In the

following, we shall study the waveform formula from different point of view because of its importance. Namely we shall use the Green's function for the wave operator in the Coulomb type potential generated by the mass of the source. This method is straightforward and enables us to see explicitly how the tail term originates from the difference between the flat light cone and the true one, which is due to the mass of the source GM/c^2 as the lowest order correction. Schäfer [17] and Nakamura [18] have obtained the quadrupole energy loss formula including the contribution from the tail term by studying the wave propagation in the Coulomb type potential.

This paper is organized as follows. In section 2, we derive the formula for the tail of gravitational waves by using the Green's function in the Coulomb type potential. In section 3, we discuss its relation with previous works. We use the units of $c = G = 1$.

II. GRAVITATIONAL WAVES IN COULOMB TYPE POTENTIAL

We wish to clarify that the main part of the tail term is due to the propagation of gravitational waves on the light cone which deviates slightly from the flat light cone owing to the mass of the source. For this purpose, we shall work in the harmonic coordinate since the deviation may be easily seen in the reduced Einstein's equation in this coordinate.

$$(\eta^{\alpha\beta} - \bar{h}^{\alpha\beta})\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\Theta^{\mu\nu} + \bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}, \quad (2.1)$$

where $\eta^{\mu\nu}$ is the Minkowskian metric, $\bar{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}$ and

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}). \quad (2.2)$$

Here, $t_{LL}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor [19]. Since we look for $1/r$ part of the solution, the spatial derivative is not relevant in the differential operator. Therefore, it is more convenient for our purpose to transform Eq.(2.1) into the following form

$$(\square - \bar{h}^{00}\partial_0\partial_0)\bar{h}^{\mu\nu} = -16\pi\mathcal{S}^{\mu\nu}, \quad (2.3)$$

where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and we defined $\mathcal{S}^{\mu\nu}$ as

$$\mathcal{S}^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{16\pi}(\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha} + 2\bar{h}^{0i}\bar{h}^{\mu\nu}{}_{,0i} + \bar{h}^{ij}\bar{h}^{\mu\nu}{}_{,ij}). \quad (2.4)$$

Substituting the lowest order expression for \bar{h}^{00} , we obtain our basic equation

$$\square_M \bar{h}^{\mu\nu} = \tilde{\tau}^{\mu\nu}, \quad (2.5)$$

where the effective source $\tilde{\tau}^{\mu\nu}$ is defined as

$$\tilde{\tau}^{\mu\nu} = \mathcal{S}^{\mu\nu} - \frac{1}{16\pi} \left(\bar{h}^{00} - \frac{4M}{r} \right) \bar{h}^{\mu\nu}{}_{,00}, \quad (2.6)$$

Here, M is the total mass of the source and \square_M is defined as

$$\square_M = \left[- \left(1 + \frac{4M}{r} \right) \frac{\partial^2}{\partial t^2} + \Delta \right], \quad (2.7)$$

where Δ is the Laplacian in the flat space. In case of the source with strong internal gravity like binary neutron stars, M may be regarded as the ADM mass [24], since $1/r$ part of \bar{h}^{00} in the harmonic coordinate becomes the ADM mass. In this way, our derivation may be applied to such compact sources.

The solution for (2.5) may be written down by using the retarded Green's function in the following form

$$\bar{h}^{\mu\nu}(x) = \int d^4x' G_M^{(+)}(x, x') \tilde{\tau}^{\mu\nu}(x'). \quad (2.8)$$

The retarded Green's function is defined as satisfying the equation;

$$\square_M G_M^{(+)}(x, y) = \delta^4(x - y), \quad (2.9)$$

with an appropriate boundary condition.

The Green's function satisfying Eq.(2.9) can be constructed by using the homogeneous solutions for the equation

$$\square_M \Psi = 0. \quad (2.10)$$

The homogeneous solution for Eq.(2.10) takes a form of

$$e^{-i\omega t} f_l(\rho) Y_{lm}(\theta, \phi), \quad (2.11)$$

where we defined

$$\rho = \omega r. \quad (2.12)$$

Then the radial function $\tilde{f}_l(\rho) \equiv \rho f_l(\rho)$ satisfies

$$\left(\frac{d^2}{d\rho^2} + 1 + \frac{4M\omega}{\rho} - \frac{l(l+1)}{\rho^2} \right) \tilde{f}_l(\rho) = 0, \quad (2.13)$$

so that Eq.(2.11) is a solution for Eq.(2.10). Thus we can obtain homogeneous solutions for Eq.(2.10) by choosing $\tilde{f}_l(\rho)$ as one of spherical Coulomb functions; $u_l^{(\pm)}(\rho; \gamma)$ and $F_l(\rho; \gamma)$ with $\gamma = -2M\omega$. Here, we adopted the following definition of the spherical Coulomb function [20] as

$$\begin{aligned} F_l(\rho; \gamma) &= c_l e^{i\rho} \rho^{l+1} F(l+1+i\gamma|2l+2|-2i\rho), \\ u_l^{(\pm)}(\rho; \gamma) &= \pm 2i e^{\mp i\sigma_l} c_l e^{\pm i\rho} \rho^{l+1} W_1(l+1 \pm i\gamma|2l+2| \mp 2i\rho), \end{aligned} \quad (2.14)$$

where c_l and σ_l are defined as

$$\begin{aligned} c_l &= 2^l e^{-\pi\gamma/2} \frac{|\Gamma(l+1+i\gamma)|}{(2l+1)!}, \\ \sigma_l &= \arg\Gamma(l+1+i\gamma). \end{aligned} \quad (2.15)$$

Here, F and W_1 are the confluent hypergeometric function and the Whittaker's function respectively. These spherical Coulomb functions have asymptotic behavior as

$$u_l^{(\pm)} \sim \exp\left[\pm i\left(\rho - \gamma \ln 2\rho - \frac{1}{2}l\pi\right)\right] \quad \text{for } r \rightarrow \infty, \quad (2.16)$$

and

$$F_l \sim c_l \rho^{l+1} \quad \text{for } r \rightarrow 0. \quad (2.17)$$

Thus we obtain the retarded Green's function in the following form.

$$\begin{aligned} G_M^{(+)}(x, x') &= \sum_{lm} e^{i\sigma_l} \int d\omega \operatorname{sgn}(\omega) \left(\Psi^{+\omega lm}(x) \Psi^{S\omega lm*}(x') \theta(r-r') \right. \\ &\quad \left. + \Psi^{S\omega lm}(x) \Psi^{+\omega lm*}(x') \theta(r'-r) \right), \end{aligned} \quad (2.18)$$

where we defined $\Psi^{+\omega lm}(x)$ and $\Psi^{S\omega lm}(x)$ as

$$\begin{aligned}\Psi^{+\omega lm}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} u_l^{(+)}(\rho; \gamma) Y_{lm}, \\ \Psi^{S\omega lm}(x) &= \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} F_l(\rho; \gamma) Y_{lm}.\end{aligned}\tag{2.19}$$

For slow motion sources, we evaluate the asymptotic form of the Green's function up to $O(M\omega)$ as

$$\begin{aligned}G_M^{(+)}(x, x') &= \frac{1}{r} \sum_{lm} \frac{(-i)^l}{2\pi(2l+1)!!} \int d\omega \left\{ 1 + \pi M\omega + 2iM\omega \left(\ln 2M\omega - \sum_{s=1}^l \frac{1}{s} + C \right) \right. \\ &\quad \left. + O(M^2\omega^2) \right\} e^{-i\omega(t-r-t')} (\omega r')^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') + O(r^{-2}) \\ &= \frac{1}{r} \sum_{lm} \frac{(-i)^l}{2\pi(2l+1)!!} \int d\omega \left[1 + 2M\omega \left\{ i \left(- \sum_{s=1}^l \frac{1}{s} + \ln 2M \right) + \frac{\pi}{2} \text{sgn}(\omega) \right. \right. \\ &\quad \left. \left. + i(\ln |\omega| + C) \right\} + O(M^2\omega^2) \right] e^{-i\omega(t-r-t')} (\omega r')^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') \\ &\quad + O(r^{-2}),\end{aligned}\tag{2.20}$$

where C is Euler's number. In deriving the above expression, we have used Eqs.(2.16) and (2.17) and the following expansion for c_l and σ_l in $M\omega$

$$\begin{aligned}c_l &= \frac{1 + \pi M\omega + O(M^2\omega^2)}{(2l+1)!!}, \\ \sigma_l &= 2M\omega \left(C - \sum_{s=1}^l \frac{1}{s} \right) + O(M^2\omega^2).\end{aligned}\tag{2.21}$$

Now we apply the formula [21,22]

$$\omega \int_0^1 dv e^{i\omega v} \ln v + i \int_1^\infty \frac{dv}{v} e^{i\omega v} = -\frac{\pi}{2} \text{sgn}(\omega) - i(\ln |\omega| + C),\tag{2.22}$$

to Eq.(2.20), then we obtain

$$\begin{aligned}G_M^{(+)}(x, x') &= \frac{1}{r} \sum_{lm} \frac{(-i)^l}{2\pi(2l+1)!!} \int d\omega \left\{ 1 - 2M \left(- \sum_{s=1}^l \frac{1}{s} + \ln 2M \right) \frac{d}{dt} + 2M \left(\int_0^1 dv e^{i\omega v} \ln v \right) \frac{d^2}{dt^2} \right. \\ &\quad \left. + 2M \left(\int_1^\infty \frac{dv}{v} e^{i\omega v} \right) \frac{d}{dt} + O(M^2\omega^2) \right\} \\ &\quad \times e^{-i\omega(t-r-t')} (\omega r')^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') + O(r^{-2}).\end{aligned}\tag{2.23}$$

Here, we assume the no incoming radiation condition on the initial hypersurface so that we may take

$$\lim_{v \rightarrow \infty} e^{-i\omega(t-r-v-t')} \ln v \rightarrow 0. \quad (2.24)$$

Thus we can make the following replacement

$$\int_1^\infty \frac{dv}{v} e^{i\omega v} \rightarrow -i\omega \int_1^\infty dv e^{i\omega v} \ln v = \left(\int_1^\infty dv e^{i\omega v} \ln v \right) \frac{d}{dt}. \quad (2.25)$$

Inserting Eq.(2.25) into Eq.(2.23), we finally obtain the desired expression for the retarded Green's function

$$\begin{aligned} G_M^{(+)}(x, x') &= \frac{1}{r} \sum_{lm} \frac{(-i)^l}{2\pi(2l+1)!!} \int d\omega \left[1 + 2M \left(\sum_{s=1}^l \frac{1}{s} - \ln 2M \right) \frac{d}{dt} \right. \\ &\quad \left. + 2M \left(\int_0^\infty dv e^{i\omega v} \ln v \right) \frac{d^2}{dt^2} + O(M^2 \omega^2) \right] \\ &\quad \times e^{-i\omega(t-r-t')} (\omega r')^l Y_{lm}(\Omega) Y_{lm}^*(\Omega') + O(r^{-2}) \\ &= \frac{1}{r} \text{ part of} \\ &\quad \left[G_0(x, x') + 2M \frac{d^2}{dt^2} \sum_{lm} \int dv \left\{ \ln \left(\frac{v}{2M} \right) + \sum_{s=1}^l \frac{1}{s} \right\} G_0^{lm}(t-v, \mathbf{x}, x') \right. \\ &\quad \left. + O(M^2) \right] + O(r^{-2}), \end{aligned} \quad (2.26)$$

where we defined the spherical harmonic expansion coefficient of the flat Green's function as

$$G_0^{lm}(x, x') = Y_{lm}(\Omega') \int d\Omega' G_0(x, x') Y_{lm}(\Omega'). \quad (2.27)$$

As a result, we obtain the waveform generated by the effective source as

$$\begin{aligned} h_{ij}^{TT} &= \frac{4}{r} P_{ijpq} \sum_{l=2}^\infty \frac{1}{l!} \left[n_{L-2} \left\{ \tilde{M}_{pqL-2}(t-r) + 2M \frac{d^2}{dt^2} \int dv \left\{ \ln \left(\frac{v}{2M} \right) + \sum_{s=1}^{l-2} \frac{1}{s} \right\} \right. \right. \\ &\quad \left. \left. \times \tilde{M}_{pqL-2}(t-r-v) \right\} - \frac{2l}{l+1} n_{aL-2} \left\{ \epsilon_{ab(p} \tilde{S}_{q)bL-2}(t-r) \right. \right. \\ &\quad \left. \left. + 2M \frac{d^2}{dt^2} \int dv \left\{ \ln \left(\frac{v}{2M} \right) + \sum_{s=1}^{l-1} \frac{1}{s} \right\} \epsilon_{ab(p} \tilde{S}_{q)bL-2}(t-r-v) \right\} \right. \\ &\quad \left. + O(M^2) \right] + O(r^{-2}), \end{aligned} \quad (2.28)$$

where P_{ijpq} is the transverse and traceless projection tensor and parentheses denote symmetrization, and n_L is the tensor product of L radial unit vectors. Although this waveform

corresponds to (C1) by Blanchet [23], \tilde{M}_{pqL-2} and \tilde{S}_{pqL-2} are the mass and current multipole moments generated by the full nonlinear effective source $\tilde{\tau}^{\mu\nu}$. They take the same forms as in [14], but here $\tilde{\tau}^{\mu\nu}$ is used in place of $\tau^{\mu\nu}$ in [14], where

$$\tau^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{16\pi} \left(\bar{h}^{\mu\alpha}{}_{,\beta} \bar{h}^{\nu\beta}{}_{,\alpha} + \bar{h}^{\alpha\beta} \bar{h}^{\mu\nu}{}_{,\alpha\beta} \right). \quad (2.29)$$

It is worthwhile to point out that $\ln 2M$ in Eq.(2.28) can be removed by using the freedom to time translation.

III. COMPARISON WITH THE PREVIOUS WORK

A. waveform in the post-Minkowskian approximation

By using the post-Minkowskian approximation, Blanchet obtained the general formula for gravitational waves including tail [23],

$$h_{ij}^{TT} = \frac{4}{r} P_{ijpq} \sum_{l=2}^{\infty} \frac{1}{l!} \left[n_{L-2} U_{pqL-2}(t-r) - \frac{2l}{l+1} n_{aL-2} \epsilon_{ab(p} V_{q)bL-2}(t-r) \right] + O(r^{-2}), \quad (3.1)$$

where the radiative mass moment and radiative current moment are defined as

$$U_L^{(l)}(u) = M_L^{(l)}(u) + 2GM \int_0^{\infty} dv M_L^{(l+2)}(u-v) \left\{ \ln\left(\frac{v}{P}\right) + \kappa_l \right\} + O(G^2 M^2), \quad (3.2)$$

$$V_L^{(l)}(u) = S_L^{(l)}(u) + 2GM \int_0^{\infty} dv S_L^{(l+2)}(u-v) \left\{ \ln\left(\frac{v}{P}\right) + \kappa'_l \right\} + O(G^2 M^2). \quad (3.3)$$

Here the moments M_L and S_L do not contain the nonlinear contribution outside the matter, P is a constant with temporal dimension, and κ_l and κ'_l are defined as

$$\kappa_l = \sum_{s=1}^{l-2} \frac{1}{s} + \frac{2l^2 + 5l + 4}{l(l+1)(l+2)}, \quad (3.4)$$

and

$$\kappa'_l = \sum_{s=1}^{l-1} \frac{1}{s} + \frac{l-1}{l(l+1)}. \quad (3.5)$$

In black hole perturbation, the same tail corrections with multipole moments induced by linear perturbations have been obtained [25].

Equation (2.28) does not apparently agree with Eqs.(3.2) and (3.3), because definitions of the moments $\{\tilde{M}_L, \tilde{S}_L\}$ and $\{M_L, S_L\}$ are different. We shall show below the equivalence between our expression and that of Blanchet [23], by counting the contribution from nonlinear terms like $M \times M_L$ or $M \times S_L$ in the source $\tilde{\tau}^{\mu\nu}$.

B. Contributions from the nonlinear sources

In order to evaluate the tail of the waveform produced by the nonlinear sources in $\tilde{\tau}^{\mu\nu}$, it is enough to use the flat Green's function

$$G_0(x, x') = -i \sum_{lm} \int d\omega \frac{\omega}{\pi} \left(e^{-i\omega t} h_l(\omega r) Y_{lm}(\Omega) e^{i\omega t'} j_l(\omega r') Y_{lm}^*(\Omega') \theta(r - r') \right. \\ \left. + e^{-i\omega t} j_l(\omega r) Y_{lm}(\Omega) e^{i\omega t'} h_l(\omega r') Y_{lm}^*(\Omega') \theta(r' - r) \right), \quad (3.6)$$

where j_l and h_l are the first kind of the spherical Bessel function and the spherical Hankel function, respectively.

We consider the asymptotic form of the following retarded integral

$$\square^{-1} \left[\frac{\hat{n}_L}{r^k} F(t - r) \right] = \int d^4 x' G_0(x, x') \left[\frac{\hat{n}'_L}{r'^k} F(t' - r') \right] \\ \rightarrow -i \int d\omega \omega e^{-i\omega t} h_l(\omega r) F_\omega \hat{n}_L \int_0^r r'^2 dr' j_l(\omega r') \frac{e^{i\omega r'}}{r'^k} \\ \text{for large } r, \quad (3.7)$$

where the hat denotes the symmetric and traceless part of tensor products and we defined F_ω as

$$F_\omega = \frac{1}{2\pi} \int dt e^{i\omega t} F(t). \quad (3.8)$$

Thus, the contribution from nonlinear sources can be evaluated by using the formula

$$\square^{-1} \left[\frac{\hat{n}_L}{r^k} F(t - r) \right] = -2^{l+1} \lim_{\lambda \rightarrow 0} \left(\sum_{n=0}^{\infty} \frac{(l+n+1)! \Gamma(-k+l+3+2n-\lambda)}{n! (2l+2n+2)!} \right) \\ \times \frac{\hat{n}_L}{r} \overset{(k-3)}{F}(t - r) + O(r^{-2}), \quad (3.9)$$

where we used

$$j_l(z) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-)^n (z/2)^{l+2n}}{n! \Gamma(l+n+3/2)}. \quad (3.10)$$

Since we consider both $M \times M_L$ and $M \times S_L$ in the nonlinear source, we have to treat the following type of retarded integral

$$\begin{aligned} \square^{-1} \left[\partial_i \left(\frac{1}{r} \right) \hat{\partial}_Q \left(\frac{F(t-r)}{r} \right) \right] &= (-)^{q+1} \sum_{j=0}^q \frac{(q+j)!}{2^j j! (q-j)!} \\ &\times \square^{-1} \left[\left(\hat{n}_{iQ} + \frac{q}{2q+1} \delta_{i < a_q} \hat{n}_{Q-1} \right) \frac{1}{r^{j+3}} \frac{(q-j)}{F}(t-r) \right]. \end{aligned} \quad (3.11)$$

Using Eq.(3.9), we obtain

$$\begin{aligned} &\square^{-1} \left[\partial_i \left(\frac{1}{r} \right) \hat{\partial}_Q \left(\frac{F(t-r)}{r} \right) \right] \\ &= \frac{(-)^q}{2(q+1)} \left[\hat{n}_{iQ} - \frac{q+1}{2q+1} \delta_{i < a_1} \hat{n}_{Q-1} \right] \frac{1}{r} \frac{(q)}{F}(t-r) + O(r^{-2}), \end{aligned} \quad (3.12)$$

where, in the limit $\lambda \rightarrow 0$ of Eq.(3.12), poles of gamma functions cancel out in total and the finite values are obtained. This is same with the expression (C2) obtained by Blanchet [23].

Applying Eqs.(3.12) to the nonlinear source $\tilde{\tau}^{\mu\nu}$, we can evaluate its contribution to the waveform. Together with Eq.(2.28), we obtain the waveform in the same form as Eq.(3.1).

IV. DISCUSSION

By considering the propagation of gravitational waves generated by slow motion sources in the Coulomb type potential, we have derived the formula (2.28) for gravitational waves including tail. Our equation (2.28) has been shown to be same as Eqs.(3.2)-(3.5) in the previous work [23].

We would like to emphasize two points on Eq.(2.28): First, in deriving Eq.(2.28), spherical Coulomb functions are used, since we use the wave operator (2.7) which take account of the Coulomb-type potential M/r . As a consequence, $\ln(v/2M)$ appears naturally

in Eq.(3.1). This is in contrast with Blanchet and Damour's method, where an arbitrary constant with temporal dimension P appears in the form of $\ln(v/P)$. Our derivation shows that the main part of the tail, which needs the past history of the source only through $\ln v$, is produced by propagation in the Coulomb-type potential. The reason for saying the main part of the tail is as follows: Only the *log* term has a hereditary property expressed as the integral over the past history of the source, since the constants κ_l and κ'_l represent merely instantaneous parts after performing the integral under the assumption that the source approaches static as the past infinity. Our method allows us to study the effect on the waveform formula by the modification of the Coulomb type potential near the material source. This effect might be important to make the comparison between theory and observation. We shall discuss this in future.

The second point relates with physical application: At the starting point of our derivation, the Fourier representation in frequency space has been used. Such a representation seems to simplify the calculation of gravitational waveforms from compact binaries in the quasi-circular orbit, since such a system can be described by a characteristic frequency. Applications to physical systems will be also done in the future.

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